ULTRA-UNIVERSAL MODELS, THEORIES AND ELEMENTARY CLASSES

by

Colin Naturman

and

Henry Rose

RR 073

APRIL 1989

RESEARCH REPORTS

DEPARTMENT OF MATHEMATICS

THE UNIVERSITY OF CAPE TOWN

ULTRA-UNIVERSAL MODELS, THEORIES AND ELEMENTARY CLASSES

by

Colin Naturman

and

Henry Rose

RR 073

APRIL 1989

An expanded version of this paper has subsequently been published as the journal article "Naturman, Colin, and Henry Rose. Ultra-universal models. Quaestiones Mathematicae 15.2 (1992): 189-195." available at https://doi.org/10.1080/16073606.1992.9631683

RESEARCH REPORTS

DEPARTMENT OF MATHEMATICS

THE UNIVERSITY OF CAPE TOWN

Printed in the Department of Mathematics, University of Cape Town, Private Bag, Rondebosch 7700, Republic of South Africa.

April 1989

ULTRA-UNIVERSAL MODELS, THEORIES AND ELEMENTARY CLASSES

Colin Naturman

and

Henry Rose[1]

Abstract

If K is an elementary class of models of a theory Σ, then K and Σ are ultra-universal if there is an $M \in K$ such that every $N \in K$ is embeddable in an ultrapower of M. A syntactic characterization of ultra-universal theories is given, and examples and counterexamples of ultra-universal theories are considered. It is shown that not all varieties are ultra-universal. Sufficient conditions for an elementary class to be ultra-universal are considered.

1980 Mathematics Subject Classification 03C20 03C50
03C52 03C05

---

[1] The work of the second author was supported by a grant from the University of Cape Town Research Committee, and by the Topology Research Group from the University of Cape Town and the South African Council for Scientific and Industrial Research.

INTRODUCTION

The concept of an ultra–universal algebra in a variety was introduced by Bruyns and Rose in [1]. In this paper we generalize this concept to models of a first order theory. We also define the concepts of ultra–universal theories and ultra–universal elementary classes. Ultra–universal members of the elementary class of all non–trivial closure algebras were already investigated by McKinsey and Tanski as 'absolutely free' closure algebras. (See Example 3.10.)

In section 1 we give characterizations of ultra–universal theories and models. We consider counterexamples and examples of ultra–universal theories in section 2 and 3 respectively. We conclude the paper by stating some open problems.

NOTATIONAL CONVENTIONS

$\mathscr{L}$ will denote a first order language with equality. By a theory we simply mean a set of sentences in such a language (not necessarily a closed set of sentences). If K is a class of models for $\mathscr{L}$ then Th K will denote the theory of K, i.e. the set of all sentences of $\mathscr{L}$ which hold in all members of K.

$\forall$ will denote the set of universal sentences of the language $\mathscr{L}$ concerned.

1. Characterizations of Ultra–universal Theories and Models.

Definition 1.1 (cf. Bruyns & Rose [2])

Let Σ be a theory. A model M of Σ is called an *ultra–universal* model of Σ iff every model of Σ is embeddable in an ultrapower of M .

If Σ has an ultra–universal model we say that Σ is *ultra–universal.* If K is an elementary class we say that K is *ultra–universal* iff Th K is ultra–universal. An ultra–universal model of Th K is said to be *ultra–universal in* K . □

The definition of an ultra–universal model of a theory Σ refers only to structural properties of the models of Σ . However we can immediately obtain a semantic characterization of ultra–universal models:

Corollary 1.2

Let M be a model of Σ . The following are equivalent:

i) M is an ultra–universal model of Σ .

ii) Every universal sentence holding in M holds in all models of Σ .

iii) Every existential sentence holding in some model of Σ holds in M .

(See [1] pp. 185–188.) □

Obviously any complete consistent theory is ultra–universal.

Let Σ be a consistent closed theory. Then Σ has a model M and Th{M} is then a complete consistent closed extension of Σ. Moreover any complete consistent closed extension Σ' of Σ is of this form. Corollary 1.2 tells us that Σ is ultra–universal iff Σ' can be chosen so that $\Sigma' \cap \forall = \Sigma \cap \forall$.

We now show that ultra–universal theories can be characterized syntactically.

Theorem 1.3

The following are equivalent:

i) Σ is ultra–universal.

ii) Σ is consistent and for all universal sentences φ and ψ, $\Sigma \vdash \varphi \vee \psi$ implies $\Sigma \vdash \varphi$ or $\Sigma \vdash \psi$.

Proof

(i) ⇒ (ii):

Assume (i). Let M be an ultra–universal model of Σ. Since Σ has a model it is consistent. Let φ, ψ be universal sentences such that $\Sigma \vdash \varphi \vee \psi$. Then $M \models \varphi \vee \psi$ and so $M \models \varphi$ or $M \models \psi$, say $M \models \varphi$. Then for all models N of Σ, $N \models \varphi$, and so $\Sigma \vdash \varphi$. Similarly if $M \models \psi$ we get $\Sigma \vdash \psi$.

(ii) ⇒ (i):

Assume (ii). Put $\Gamma = \{\neg\sigma : \sigma \in \forall \text{ and } \Sigma \nvdash \sigma\}$. We show that $\Sigma \cup \Gamma$ is consistent. Let Δ be a finite subset of $\Sigma \cup \Gamma$. If $\Delta \subseteq \Sigma$ then obviously Δ is consistent, so suppose there is an $n < \omega$ and $\sigma_0, \ldots, \sigma_n \in \forall$ such that $\Sigma \nvdash \sigma_i$ for all $i \in \{0, \ldots, n\}$ and $\Delta \cap \Gamma = \{\neg\sigma_0, \ldots, \neg\sigma_n\}$. Put $\sigma = \sigma_0 \vee \ldots \vee \sigma_n$. Suppose $\Sigma \vdash \sigma$. Then by applying (ii) we get $\Sigma \vdash \sigma_i$ for some $i \in \{0, \ldots, n\}$ – a contradiction. Hence $\Sigma \nvdash \sigma$ and so there is a model N of Σ such that $N \not\models \sigma$. Then $N \models \neg\sigma$. Now $\neg\sigma \equiv \neg\sigma_0 \wedge \ldots \wedge \neg\sigma_n$ and so N is a model of Δ. Hence Δ is consistent. By compactness $\Sigma \cup \Gamma$ is consistent and so it has a model M. Then in particular M is a model of Σ. We show that M is an ultra–universal model of Σ. Let φ be a universal sentence such that $M \models \varphi$. Suppose $\Sigma \nvdash \varphi$. Then $\neg\varphi \in \Gamma$ and so $M \models \neg\varphi$, a contradiction. Hence $\Sigma \vdash \varphi$ and so $N \models \varphi$ for all models of Σ, as required.

We can now characterize ultra–universal theories in terms of filters in a certain sublattice of the Lindenbaum algebra of the language $\mathscr{L}$.

Let $\mathbf{B} = \langle B, \wedge, \vee, \neg, F, T\rangle$ be the Lindenbaum algebra of $\mathscr{L}$. Let $[\varphi]$ denote the equivalence class of a formula φ under the equivalence relation $\equiv$ given by $\varphi \equiv \psi$ iff $\vdash(\varphi \leftrightarrow \psi)$.

Put $A = \{[\varphi]: \varphi \in \forall\}$. F is the equivalence class of all universally false sentences and T is the equivalence class of all universally true sentences. Hence $F = [(\forall x)(x \neq x)]$ and $T = [(\forall x)(x = x)]$ and so we see that $F, T \in A$. Also A is closed under $\vee$ and $\wedge$. Hence we obtain a distributive 0,1–lattice (i.e. a lattice with distinguished bottom and top elements) $\mathbf{A} = \langle A, \wedge, \vee, F, T\rangle$.

Given a theory Σ in $\mathscr{L}$ put $\mathscr{D}_\Sigma = \{[\varphi]: \Sigma \vdash \varphi\}$. Then $\mathscr{D}_\Sigma$ is a filter in $\mathbf{B}$. Moreover all filters in $\mathbf{B}$ are of this form, since if $\mathscr{D}$ is a filter in $\mathbf{B}$ putting $\Sigma = \cup\mathscr{D}$ gives $\mathscr{D}_\Sigma = \mathscr{D}$.

$\mathscr{D}_\Sigma$ is proper iff Σ is consistent. $\mathscr{D}_\Sigma$ is an ultrafilter iff Σ is complete and consistent.

Now given a theory Σ, $\mathscr{D}_\Sigma \cap A$ is a filter in $\mathbf{A}$. From Theorem 1.3 and the remark after corollary 1.3 we obtain:

Corollary 1.4

The following are equivalent:

i) Σ is ultra–universal.

ii) $\mathcal{D}_\Sigma$ can be extended to an ultrafilter $\mathcal{D}$ in B with $\mathcal{D} \cap A = \mathcal{D}_\Sigma \cap A$.

iii) $\mathcal{D}_\Sigma \cap A$ is a prime proper filter in the 0,1–lattice A. □

2. Non–ultra–universal Theories.

Before looking at examples of ultra–universal theories we show that not all theories are ultra–universal.

Example 2.1

Let $\Sigma = \{(\forall x)(x^2 = e) \vee (\forall x)(x^3 = e)\}$ in the language for groups. The trivial group is a model of Σ so Σ is consistent. However $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are both models of Σ and so we see that $\Sigma \nvdash (\forall x)(x^2 = e)$ and $\Sigma \nvdash (\forall x)(x^3 = e)$. Hence Σ cannot be ultra–universal.

We can obviously generalize this result: Let $\Sigma = \{(\forall x)(x^m = e) \vee (\forall x)(x^n = e)\}$ where $n,m \geq 2$ are relatively prime. Then Σ is not ultra–universal. □

In particular we see that even if a theory has universal axioms it may fail to be ultra–universal. In [2] and [5] it is shown that the varieties of groups, lattices and monoids are ultra–universal. The following result shows however that not all varieties are ultra–universal.

Theorem 2.2

Let $\mathcal{V}$ be a variety of algebras which have two nullary operations a and b. Suppose $\mathcal{V}$ does not satisfy the identity $(a = b)$. Then $\mathcal{V}$ is not ultra–universal.

Proof:

Suppose $\mathcal{V}$ is ultra–universal. All members of $\mathcal{V}$ satisfy $(a = b) \vee \neg(a = b)$. Since $\mathcal{V}$ does not satisfy $(a = b)$ we have by Theorem 1.3 that $\mathcal{V}$ satisfies $\neg(a = b)$. All varieties contain a trivial member. Let $N \in \mathcal{V}$ be trivial. Since $N \in \mathcal{V}$, $N \vdash \neg(a = b)$ and since N is trivial $N \vdash (a = b)$, a contradiction. Hence $\mathcal{V}$ cannot be ultra–universal.

In particular the above theorem shows that the variety of all Boolean algebras is not ultra–universal. However as we will see later (Example 3.2) the elementary class of all non–trivial Boolean algebras is ultra–universal.

3. Examples of Ultra–universal Elementary Classes and Models

We seek sufficient conditions to ensure that a elementary class is ultra–universal.

Theorem 3.1

Let K be an elementary class of algebras. Suppose there is an algebra $M \in K$ such that every finitely generated subalgebra of a member of K is embeddable in M. Then M is ultra–universal in K.

Proof:

Let $N \in K$. Let $\{N_i : i \in I\}$ be the collection of all finitely generated subalgebras of N. Every algebra is embeddable in an ultraproduct of its finitely generated sub-algebras (see [3] page 213) so there is an ultrafilter $\mathscr{D}$ over I such that N is embeddable in $\Pi_{\mathscr{D}} N_i$. Then for all $i \in I$ N_i is embeddable in M. Hence N is embeddable in $\Pi_{\mathscr{D}} M$.

The above result is essentially what whas used to show that infinite symmetric groups and function monoids are ultra-universal in the varieties of groups and monoids respectively (see [2] and [5].)

Example 3.2

Let K be an elementary class of non-trivial Boolean algebras which contains an infinite member M. All finitely generated Boolean algebras are finite and all non-trivial finite Boolean algebras are embeddable in M, so by Theorem 3.1 we see that M is ultra-universal in K.

Example 3.3

Let X be a non-empty set and let M be the semigroup of all functions with domain and co-domain X. (The operation in M is composition of functions.) M has an identity element id_x, the identity function on X. (M, id_x) is then a monoid, the function monoid on X.

If X is infinite then every countable monoid is embeddable in (M, id_x) and so as was shown in [5] (M, id_x) is ultra-universal in the variety of all monoids. Now if N is a semigroup, we can obtain a semigroup with an identity element by letting $e \notin N$ and extending the operation on N to $N \cup \{e\}$ so that e is an identity element. Then the inclusion map of N is $N \cup \{e\}$ is an embedding. Suppose N is countable. Then the monoid $(N \cup \{e\}, e)$ is embeddable in (M, id_x). Hence $N \cup \{e\}$ is embeddable in M. By Theorem 3.1 we see that M is ultra-universal in the variety of all semigroups. □

Definition 3.4

Let K be a non-empty class of structures of a fixed type.

Let $\{N_i : i \in I\}$ be a non-empty family of structures in K. If $j \in I$ and $f : N_j \to \prod_{i \in I} N_i$ is an embedding we say that f is a *factor embedding.* If in addition f satisfies the condition $\pi_j f(a) = a$ for all $a \in N_j$ (where $\pi_j : \prod_{i \in I} N_i \to N_j$ is the j^{th} projection), we say that f is a *strong factor embedding.* K is said to be *factor embeddable* iff for every non-empty family $\{N_i : i \in I\}$ of structures in K, and for all $j \in I$ there is a factor embedding $f_j : N_j \to \prod_{i \in I} N_i$. If these factor embeddings can be chosen to be strong factor embeddings then we say that K is *strongly factor embeddable.* □

The seemingly stronger condition of *strong factor embeddability* is in fact equivalent to *factor embeddability* :

Theorem 3.5

Let K be a non-empty class of structures. Then the following are equivalent:

i) K is strongly factor embeddable.

ii) K is factor embeddable.

iii) For all $M, N \in K$ there is a homomorphism from M to N.

Proof:

Obviously (i) implies (ii). Assume (ii). Let $M, N \in K$. Then there is a factor embedding $f : M \to M \times N$. Let $\pi : M \times N \to N$ be the projection onto N. Then πf is a homomorphism from M to N. So (ii) implies (iii).

Now assume (iii). Then for all $M, N \in K$ there is a homomorphism $f_{M,N}:M \to N$. Let $\{N_i : i \in I\}$ be a non–empty family of structures in K. Let $j \in I$. Define $f_j : N_j \to \prod_{i\in I} N_i$ by :

For all $a \in N_j$ $f_j(a) = (b_i)_{i\in I}$ where $b_j = a$ and $b_i = f_{N_j,N_i}(a)$ for $i \neq j$. Then f_j is a strong factor embedding. Hence (iii) implies (i). □

A useful corollary to the above is:

Corollary 3.6

Let K be a non–empty class of structures and suppose there is a structure R which is a retract of all members of K. Then K is factor embeddable.

Proof:

Let $M,N \in K$. Then there is an epimorphism $p:M \to R$ and an embedding $j:R \to M$. Then $jp:N \to M$ is a homomorphism. By Theorem 3.5 K is factor embeddable. □

Example 3.7

Let K be a non–empty elementary class of non–trivial distributive lattices. Then the two element chain is a retract of all members of K and so by the above corollary K is factor embeddable. □

Theorem 3.8

Let K be a non–empty factor embeddable elementary class of algebras. Suppose K is closed under the formation of products and finitely generated subalgebras. Then K is ultra–universal.

Proof:

Note that the collection of all finitely generated algebras in K up to isomorphism forms a set $\{N_i : i \in I\}$. Put $M = \prod_{i\in I} N_i$.

Since K is closed under products $M \in K$. Let $Q \in K$ and let N be a finitely generated subalgebra of Q . Then $N \in K$ and so there is an $i \in I$ such that $N \cong N_i$. Since K is factor embeddable N_i is embeddable in M and so N is embeddable in M. By Theorem 3.1 M is ultra–universal in K .

Example 3.9

Let $\mathscr{V}$ be a variety of algebras. Every variety has a trivial member which is unique up to isomorphism and this trivial member is an epimorphic image of all algebras in the variety. Now suppose every member of $\mathscr{V}$ has a trivial subalgebra. Then the trivial member of $\mathscr{V}$ is in fact a retract of all members of $\mathscr{V}$, and so $\mathscr{V}$ is factor embeddable. Since varieties are closed under products and subalgebras, by Theorem 3.8 $\mathscr{V}$ is ultra–universal.

In particular any subvariety of groups, monoids or lattices is ultra–universal. □

Example 3.10

In [4] McKinsey and Tarski mention 'absolutely free' closure algebras. These are in fact closure algebras which are ultra–universal in the elementary class **K** of all non–trivial closure algebras.

To summarize the results in [4]. Any totally disconnected dissectable closure algebra is ultra–universal in **K**. In particular the closure algebra over Cantor's discontinuum or any denumberable dense–in–itself topological space, is ultra–universal in **K**. □

Open Problems

Let G denote the group of all permutations of $\aleph_0$ with finite support. Is G ultra–universal in the variety of all groups ? Since $S_{\aleph_0}$ (the group of all permutations of $\aleph_0$ is ultra–universal, this is equivalent to asking if $S_{\aleph_0}$ is embeddable in an ultra power of G.

G is embeddable in an ultra product of its finitely generated subgroups which are all finite. Also any finite group is embeddable in G. Hence the above question is equivalent to: Can every group (equivalently $S_{\aleph_0}$) be embedded in an ultra product of finite groups?

This is also equivalent to asking: Is it true that for any universal sentence φ, all finite groups satisfy φ iff all groups (equivalently $S_{\aleph_0}$) satisfy φ?

Notice that the variety of all groups satisfies the amalgamation property and has a sub–directly irreducable ultra–universal member (viz. $S_{\aleph_0}$). This is also true for lattices.

(Infinite partition lattices are ultra–universal and simple, see [2].) Is there some general theorem concerning ultra–universal subdirectly irreducible algebras and amalgamation? □

REFERENCES

1. J.L. Bell and A.B. Slomson "Models and Ultraproducts" *North–Holland* (1969).
2. P. Bruyns and H. Rose "Varieties with Cofinal Sets" *Preprint.*
3. S. Burris and H.P. Sankapanavar "A Course in Universal Algebra" *Springer–Verlag* (1981).
4. J.C.C. McKinsey and A. Tarski "The Algebra of Topology" *Ann. Math* 1944 45, pp141–181.
5. C.A. Naturman and H. Rose "Elementary Equivalent Pairs of Algebras Associated with Sets" *Preprint.*

Department of Mathematics
University of Cape Town
Rondebosch 7700
Cape Town
South Africa